and hence our results would not be consistent with the f^0 [16] being on the vacuum trajectory. We also calculated the residue of the vacuum pole at $s=0$. The residue corresponding to the trajectory shown in Fig. 5 gave an asymptotic total $\pi-\pi$ cross section of 3 mb as compared to a value of the 15 mb obtained using the factorization theorem[17] and the asymptotic πN and NN cross sections.

We feel that both the problem (a) that the output ρ width is larger than the input ρ width and the problem (b) that using the input ρ parameters which yield a ρ resonance to calculate the $(I=0)$ vacuum trajectory give $\alpha_P(0)>1$ are largely due to the one channel approximation. The effect of an inelastic channel below its threshold is to, (i) always act as an attraction, and (ii) tend to narrow a resonance. Hence if we include the inelastic effects in the $I=1$ channel, which we expect to be due largely to the $\pi\omega$ channel,[1] this would narrow the output ρ width, and increase the attraction so that a somewhat smaller $\alpha_\rho^{\text{In}}(0)$ would be required.[18] On the other hand, the $\pi\omega$ channel does not couple to the $I=0$ channel so that this additional attraction would not be present and hence we would have a smaller $\alpha_P(0)$.

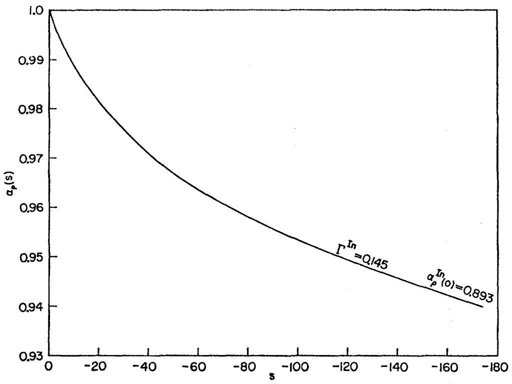

FIG. 5. The $I=0$ vacuum trajectory $\alpha_P(s)$ which has been adjusted to cross $s=0$ at $l=1$.

[16] W. Selove, V. Hagopian, H. Broad, A. Baker, and E. Leboy, Phys. Rev. Letters 9, 277 (1962).

[17] M. Gell-Mann, Phys. Rev. Letters 8, 263 (1962); V. Gribov and I. Pomeranchuk, *ibid.* 8, 343 (1962).

[18] A relatively small change in $\alpha_\rho^{\text{In}}(0)$ produces a large shift in the output resonance position.

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Method of the Self-Consistent Field in General Relativity and its Application to the Gravitational Geon*

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Concentrations of radiation held together for a long time by their own gravitational attraction ("geons") have been studied for nearly a decade. We extend the previous analyses to the case where gravitational waves are the source of the geon's mass energy. To analyze these solutions of the free-space Einstein equations with persistent features, we develop an approximation method to treat small ripples on a strongly curved background metric. The background metric describes the large-scale persistent features of the geon and is taken to be spherically symmetric. The waves superimposed on this background have an amplitude small enough so that their dynamics can be analyzed in the linear approximation; however, their wavelength is so short, and their time dependence so rapid that their energy is appreciable and produces the strongly curved background metric in which they move. The Einstein equations are investigated in this limit of short wavelength. It is found that the large-scale features of thin-shell spherical gravitational geons—in fact, of thin-shell spherical geons constructed from *any* field of zero rest mass—are identical to those of the spherical electromagnetic geons analyzed previously.

I. INTRODUCTION

THE existence of nonsingular, asymptotically Euclidian solutions of the free-space Maxwell-Einstein equations with persistent large-scale features was first pointed out by Wheeler.[1] He gave the name "geons" to these and similar concentrations of zero rest-mass fields which are held together for a long time by gravitational attraction. In addition to Wheeler's electromagnetic geons,[1,2] neutrino geons have been dis-

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[1] J. A. Wheeler, Phys. Rev. 97, 511 (1955); also see J. A. Wheeler, *Geometrodynamics* (Academic Press Inc., New York, 1962).

[2] F. J. Ernst, Phys. Rev. 105, 1662 and 1665 (1957).

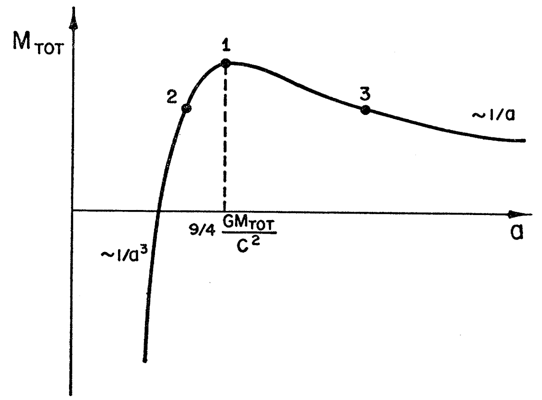

FIG. 1. Graph showing the variation of the sum of radiation energy and gravitational potential energy as the radius, a, of the active region is changed adiabatically. Under such a change the radiation remains in the same mode so that its frequency and energy vary as $1/a$; the gravitational interaction energy is always given by $G\cdot(\text{mass of radiation})^2/a$. The equilibrium geon we are discussing corresponds to position 1 at the maximum of the curve. Any small perturbation would either make the geon collapse (case 2) or disperse (case 3). In both cases the asymptotically Schwarzschildian region cannot be affected, and the total mass of the geon must remain unchanged; the field energy lost in moving along the curve is converted into "kinetic energy" of contraction or dispersion. Thus the geon in configuration 1 is at equilibrium, but unstable (as first pointed out by Wheeler, Ref. 6) with respect to change in the radius of the active region. [Instabilities of the spherical geon to decay into other (toroidal) geons have been discussed by Wheeler (Ref. 1).]

cussed.[3] Can similar geons be constructed in which gravitational waves take the place of electromagnetic or neutrino waves? Such "gravitational geons" are interesting for a number of reasons:

(a) They give further examples of concentrations of pure gravitational waves in empty space-time which represent a nonzero total mass energy. It has previously been shown[4] that certain solutions of the free-space Einstein equations represent concentrations of mass energy of gravitational origin. The previous analysis of these waves was confined to the neighborhood of an "initial" space-like surface. The solutions are everywhere regular in this neighborhood and represent typically a contracting and re-expanding pulse of gravitational radiation. In the present paper we ask whether among these there are waves which neither expand nor contract, but remain concentrated for a long time.

(b) The spherical gravitational geon provides a simple illustration of the production of large-scale curvature by small-scale ripples in the metric. The equations of general relativity, applied to a geometry far from flat—where the nonlinearities are very marked—can still be analyzed in the context of linear theory in the following remarkable sense: (i) there is a "background geometry" which is strongly curved but which is treated as static or changing only slowly with time; (ii) on this background there is superimposed a ripple in the metric which has an amplitude δg very small compared to unity and which obeys a *linear* wave equation; (iii) however, the reduced wavelength $\lambda\!\!\!^{-}$ is so short compared to the dimensions of the geon that the *effective* energy density $\sim(\delta g/\lambda\!\!\!^{-})^2$ (in cm^{-2}) or $\sim(c^4/G)(\delta g/\lambda\!\!\!^{-})^2$ (in erg/cm^3) is substantial; (iv) this effective energy density—averaged with respect to time—serves as the source which produces the curvature of the background geometry. In other words, one can use in gravitation physics the same kind of self-consistent field analysis which is already known to be so useful in atomic and nuclear physics.

(c) Geons provide well-defined models for bodies in classical general relativity. They are useful in clarifying the connection of the equations of motion in general relativity with the field equations.[5] In addition, it has recently been pointed out[6] that geons may undergo a collective motion of radial collapse (Fig. 1). Thus the long-term dynamics of a gravitational geon provide a schematic *model* for a collapsing star, free from questions about equations of state or about field equations other than those of the gravitational field. In this paper we take the initial step in such an analysis by treating the spherical gravitational geon at the equilibrium configuration (case 1 of Fig. 1).

(d) A cloud of radiation which implodes, but which has insufficient energy density to hold itself together for a prolonged period like a geon—by this token also insufficient energy density to collapse—provides an example, as yet unanalyzed in detail, of solutions of the free-space Einstein equations which remain free from singularities for all times.

An approximation procedure similar to that used by Wheeler to treat electromagnetic geons[1] is necessary to analyze the simple spherical gravitational geon. This "self-consistent field" approach is discussed in Sec. II. After recalling in Sec. III previous treatments of small ripples on a spherically symmetric background, we discuss the response of the gravitational waves to the background metric, and derive the radial equation for the amplitude of the ripples. In the region where the wave amplitude vanishes, the response of the background metric to the effective energy contained in the waves has a simple character imposed by spherical symmetry and is discussed in Sec. IV. Information about the "active" region, in which the wave amplitude differs from zero, is obtained in Sec. V. The resulting solution for the gravitational geon metric is compared and contrasted in Sec. VI with that of the electromagnetic geon.

II. "SELF-CONSISTENT FIELD" APPROACH TO TREAT SPHERICAL GRAVITATIONAL GEONS

Electromagnetic and gravitational geons can be spherically symmetric only on a time average. This is so because it is impossible to construct any truly spherically symmetric distribution of transverse tensor fields.

[3] D. R. Brill and J. A. Wheeler, Rev. Mod. Phys. **29**, 465 (1957).
[4] D. R. Brill, Ann. Phys. (N. Y.) **7**, 466 (1959).
[5] J. A. Wheeler, Rev. Mod. Phys. **33**, 63 (1961).
[6] J. A. Wheeler, Lectures at Les Houches School, Summer 1963 (to be published).

Time averaging is one of the basic tools of statistical treatments, and is widely used in general relativity. For example, in the Tolman radiation-filled universe,[7] the details of the electromagnetic field are replaced by densities of energy and pressure, obtained by averaging over a time long compared to a period of the electromagnetic waves. Only by the use of this approximation can the simple, large-scale features of the solution (such as isotropy and homogeneity) be brought to light without excessively cumbersome calculations.

Similarly, in Wheeler's treatment of the electromagnetic geon, the details of the electromagnetic field are replaced by the time average of the components of the electromagnetic stress-energy tensor. Again, this procedure allows one to treat in a simple way the large-scale features of the geon: In the approximation in which one averages over many modes of oscillation of the electromagnetic field, one obtains a stress-energy tensor, and hence a gravitational field and metric which are spherically symmetric. In such "spherical" geons any given mode of excitation is taken to propagate in the average, spherically symmetric gravitational field produced by all the rest of the radiation. This method of treating the geon may be considered an application in the field of gravitation of the well-known Hartree-Fock self-consistent approach for electrons in atoms. In both cases the method describes well the large-scale features of the object under consideration.

Separation of Metric into Background and Waves

For the gravitational geon we assume a metric of the form

$$g_{\mu\nu}=\gamma_{\mu\nu}+h_{\mu\nu}. \tag{1}$$

Here $\gamma_{\mu\nu}$ is the "background" metric which describes the long-range, time-average gravitational field of the geon, and which will generally deviate substantially from the flat-space metric; $h_{\mu\nu}$ are *small* perturbations which fluctuate rapidly in space and time, describing the local gravitational effects of the gravitational waves in the geon. In virtue of our self-consistent assumption, we consider only the time average of the long-range effects of the gravitational waves in the geon, and this time average must be spherically symmetric, in order to describe a *spherical* geon. Therefore, in a suitable system of spherical coordinates, $\gamma_{\mu\nu}$ has the well-known form[8] of a metric with these symmetry properties,

$$\gamma_{\mu\nu}=\mathrm{diag}(-e^{\nu},e^{\lambda},r^2,r^2\sin^2\theta). \tag{2}$$

Corresponding to the division (1) of the metric into a part which is spherically symmetric and a part which is small, the field equations will consist of two terms,

$$G_{\mu\nu}(g_{\alpha\beta})=G_{\mu\nu}(\gamma_{\alpha\beta})+\Delta G(\gamma_{\alpha\beta},h_{\alpha\beta})=0. \tag{3}$$

[7] R. C. Tolman, *Relativity, Thermodynamics and Cosmology* (Oxford University Press, New York, 1958), p. 427.

[8] See, for example, W. Pauli, *Theory of Relativity* (Pergamon Press, Inc., New York, 1958), p. 164.

Here $G_{\mu\nu}$ denotes the Einstein tensor formed from any metric $m_{\alpha\beta}$,

$$G_{\mu\nu}(m_{\alpha\beta})=R_{\mu\nu}(m_{\alpha\beta})-\tfrac{1}{2}m_{\alpha\beta}R(m_{\alpha\beta}) \tag{4}$$

and $\Delta G_{\mu\nu}$ is defined by Eq. (3). Equation (3) can be replaced by an equivalent set of two equations:

$$\Delta G_{\mu\nu}(\gamma_{\alpha\beta},h_{\alpha\beta})-\langle\Delta G_{\mu\nu}(\gamma_{\alpha\beta},h_{\alpha\beta})\rangle=0, \tag{4a}$$

$$G_{\mu\nu}(\gamma_{\alpha\beta})=-\langle\Delta G_{\mu\nu}(\gamma_{\alpha\beta},h_{\alpha\beta})\rangle. \tag{4b}$$

Here $\langle\ \rangle$ denotes a time average over a time long compared to the fluctuation period of the $h_{\mu\nu}$, but short compared to the time it takes light to cross the geon. Equation (4b) now shows how the $\gamma_{\alpha\beta}$ are determined by a time average of the gravitational waves, and Eq. (4a) may be regarded as a field equation for the $h_{\alpha\beta}$ in the background metric $\gamma_{\mu\nu}$.

Method of Approximation

Up to this point our Eqs. (1)–(4) do not yet imply any approximation—they are fulfilled by any solution of the Einstein equations and any static metric $\gamma_{\alpha\beta}$, the $h_{\alpha\beta}$ being then defined by Eq. (1). We must now use the assumption that the $h_{\alpha\beta}$ be small and hence approximate Eq. (4a) by its linear terms. The resulting *linear equation* will have solutions with a periodic time dependence. In the construction of the spherical geon, a superposition of such waves will be used with frequencies sufficiently high that the time average of the linear terms will vanish, $\Delta_1 G_{\mu\nu}=0$. Here we denote the order of the term in the expansion of $\Delta G_{\mu\nu}$ in powers of $h_{\mu\nu}$ and its derivatives by a subscript on the Δ (i.e., such that $\Delta G_{\mu\nu}=\sum_i \Delta_i G_{\mu\nu}$). Equation (4a) now becomes

$$\Delta_1 G_{\mu\nu}(\gamma_{\alpha\beta},h_{\alpha\beta})=0. \tag{5a}$$

The lowest order nonzero term on the right side of (4b) is of second order in $h_{\alpha\beta}$:

$$G_{\mu\nu}(\gamma_{\alpha\beta})=-\langle\Delta_2 G_{\mu\nu}(\gamma_{\alpha\beta},h_{\alpha\beta})\rangle. \tag{5b}$$

Equations (5a), (5b) are the field equations with which we shall work. Equation (5a) plays a role similar to the Maxwell equations for the electromagnetic geon. Equation (5b) has the form of the Einstein equations for $\gamma_{\alpha\beta}$ with an effective source term, which reflects the fact that $g_{\alpha\beta}$, not $\gamma_{\alpha\beta}$, is the solution of the sourceless Einstein equations.

One can simplify Eqs. (5) to involve only the Ricci tensor instead of the Einstein tensor: Contract Eqs. (5) and note that in virtue of Eq. (5b) the quantities $G_{\mu\nu}(\gamma_{\alpha\beta})$ are of the same order of magnitude as terms of second order in $h_{\alpha\beta}$. Substitute the result in Eqs. (5) to find

$$\Delta_1 R_{\mu\nu}(\gamma_{\alpha\beta},h_{\alpha\beta})=0, \tag{6a}$$

$$R_{\mu\nu}(\gamma_{\alpha\beta})=-\langle\Delta_2 R_{\mu\nu}(\gamma_{\alpha\beta},h_{\alpha\beta})\rangle. \tag{6b}$$

These equations of course also follow if one applies arguments like the above to the equations

$$R_{\mu\nu}(g_{\alpha\beta})=0$$

instead of the equivalent Eq. (3). The $G_{\mu\nu}$ form is more convenient when the equations are considered as resulting from a principle of least action, whereas the $R_{\mu\nu}$ form is more convenient for the explicit calculations of self-consistent field theory.

III. DYNAMICS OF SMALL PERTURBATIONS ON A BACKGROUND METRIC

Formulas for $\Delta R_{\mu\nu}$

In order to write out Eqs. (6) explicitly in terms of $\gamma_{\alpha\beta}$ and $h_{\alpha\beta}$, we use the expression calculated by a number of authors.[9] The result for the first- and second-order terms is:

$$\Delta_1 R_{\mu\nu}=\Delta^{\sigma}{}_{\mu\nu;\sigma}-\Delta^{\sigma}{}_{\mu\sigma;\nu}\,, \tag{7a}$$

$$\Delta_2 R_{\mu\nu}=\Lambda^{\sigma}{}_{\mu\nu;\sigma}-\Lambda^{\sigma}{}_{\mu\sigma;\nu}+\Delta^{\sigma}{}_{\mu\nu}\Delta^{\kappa}{}_{\sigma\kappa}-\Delta^{\sigma}{}_{\mu\kappa}\Delta^{\kappa}{}_{\nu\sigma}\,, \tag{7b}$$

where a semicolon denotes covariant differentiation using the background metric, and

$$\Delta^{\sigma}{}_{\mu\nu}=\tfrac{1}{2}\gamma^{\sigma\alpha}(h_{\mu\alpha;\nu}+h_{\nu\alpha;\mu}-h_{\mu\nu;\alpha})\,,$$

$$\Lambda^{\sigma}{}_{\mu\nu}=\tfrac{1}{2}h_1{}^{\sigma\alpha}(h_{\mu\alpha;\nu}+h_{\nu\alpha;\mu}-h_{\mu\nu;\alpha})\,,$$

and

$$h_1{}^{\mu\nu}=-\gamma^{\mu\alpha}\gamma^{\nu\beta}h_{\alpha\beta}\,.$$

Thus, the wave equation (6a) takes the form:

$$\gamma^{\alpha\beta}[h_{\mu\nu;\alpha\beta}+h_{\alpha\beta;\mu\nu}-h_{\mu\alpha;\nu\beta}-h_{\nu\alpha;\mu\beta}]=0\,.$$

Time and Angular Dependence of $h_{\mu\nu}$

The first step in analyzing the gravitational geon consists of solving the wave equation (6a) in a given, spherically symmetric and static background metric. The symmetry properties of the background metric imply a degeneracy in the solutions $h_{\mu\nu}$ in the following sense: For each frequency and radial dependence there are numerous solutions differing only in angular dependence, and related to each other by rotations about the center of the geon. After averaging over many modes only the time and radial dependence remain, so that we may perform our calculation of the radial wave equation on the one mode with the simplest features.

Regge and Wheeler[10] have discussed the possible angular modes of tensor waves in a spherically symmetric background. They find that the waves can be characterized by the usual quantum numbers l and m, and parity. For simplicity we shall confine attention to geons containing only odd parity modes. The form of the metric can be simplified by small coordinate transformations which change the metric to order $h_{\mu\nu}$ only. The final form of a set of odd parity modes, as given by Regge and Wheeler, is

$$\begin{aligned} h_{\mu\nu}&=r_{\mu\nu}(r)e^{-i\omega t}\Theta_l{}^m(\theta)+\text{c.c.}\,,\\ r_{\mu\nu}(r)&=h_0(r)[\delta_\mu{}^0\delta_\nu{}^3+\delta_\nu{}^0\delta_\mu{}^3]+h_1(r)[\delta_\mu{}^1\delta_\nu{}^2+\delta_\nu{}^1\delta_\mu{}^2]. \end{aligned} \tag{8}$$

Here $r_{\mu\nu}(r)$ is the radial function to be determined from Eq. (6a). The angular dependence $\Theta_l{}^m$ is determined from the spherical symmetry,

$$\Theta_l{}^m(\theta)=\sin\theta\; dY_l{}^m(\theta)/d\theta\,, \tag{9}$$

where $Y_l{}^m$ is the usual spherical harmonic.

Radial Equation

Only three nontrivial equations remain when the background metric (2) and the waves (8) are substituted into the wave equation (6a). These correspond to $(\mu,\nu)=(1,3)$, $(2,3)$, and $(0,3)$. The angular part of these equations separates off, and we obtain the three radial equations

$$i\omega e^{-\nu}\left(\frac{dh_0}{dr}+\frac{2h_0}{r}\right)+h_1\left[\frac{l(l+1)}{r^2}-\omega^2e^{-\nu}+\frac{e^{-\lambda}}{r}\left(\lambda_r-\nu_r-\frac{2}{r}\right)\right]=0\,, \tag{10a}$$

$$i\omega h_0 e^{-\nu}+e^{-\lambda}[\tfrac{1}{2}(\nu_r-\lambda_r)h_1+dh_1/dr]=0\,, \tag{10b}$$

$$\frac{d^2h_0}{dr^2}-i\omega\left[\frac{dh_1}{dr}+h_1\left(\frac{2}{r}-\frac{1}{2}(\lambda_r+\nu_r)\right)\right]-\frac{1}{2}(\lambda_r+\nu_r)\frac{dh_0}{dr}-\left[\frac{e^{\lambda}l(l+1)}{r^2}-\frac{2\nu_r}{r}\right]h_0=0\,. \tag{10c}$$

Here we have denoted some radial derivatives by subscript r's.

Since there are only two independent functions, h_0 and h_1 in the expression (8) for the gravitational wave, we need only two of the field equations (10) to determine them. The solution of any pair of field equations must also solve the remaining equations, since no physical constraint inconsistent with the symmetries was used in deriving the form (8) of the wave.

For convenience we choose the equations (10a) and (10b) since they are two simultaneous first-order equations for h_0 and h_1. One of the functions, e.g., h_0, can be eliminated to find a second-order equation for only one function,

$$\frac{d^2h_1}{dr^2}+\frac{dh_1}{dr}\left[\frac{3}{2}(\nu_r-\lambda_r)-\frac{2}{r}\right]+h_1\left[\frac{1}{2}(\nu_{rr}-\lambda_{rr})+\frac{1}{2}(\nu_r-\lambda_r)+\frac{2}{r^2}+\frac{e^{-\lambda}l(l+1)}{r^2}+k^2e^{\lambda-\nu}\right]=0\,. \tag{11}$$

[9] See, for example, T. Levi-Civita, *The Absolute Differential Calculus* (Blackie and Son, London, 1927), Chap. 8.

[10] T. Regge and J. A. Wheeler, Phys. Rev. **108**, 1063 (1957); and J. Mathews in *H. P. Robertson in Memoriam* (SIAM, Philadelphia, Pennsylvania, 1963), p. 36. Also see F. Manasse, J. Math. Phys. 4, 746 (1963) regarding errors in the paper of Regge and Wheeler. The present paper does not contain the same errors because we treat a more general case which had to be calculated independently.

To eliminate the term in dh_1/dr, define a new measure of radial distance r^* by

$$dr^* = e^{\frac{1}{2}(\lambda-\nu)}dr \qquad (12)$$

and a new measure of the field,

$$Q = e^{\frac{1}{2}(\nu-\lambda)}h_1/r. \qquad (13)$$

Then Eq. (23) becomes

$$\frac{d^2Q}{dr^{*2}} + \left[\omega^2 - \frac{e^\nu l(l+1)}{r^2} + \frac{3}{2r}\frac{d}{dr^*}(e^{\frac{1}{2}(\nu-\lambda)})\right]Q = 0. \qquad (14)$$

From a solution of this radial equation the functions h_0 and h_1 satisfying Eqs. (10) are obtained by solving Eqs. (13) and (10b):

$$h_1 = rQe^{\frac{1}{2}(\lambda-\nu)},$$
$$h_0 = (1/i\omega)d(rQ)/dr^*.$$

Equation (14) has the form of a Schrödinger wave equation: ω^2 plays the role of the energy, and the factor

$$V(r) = e^\nu l^{*2}/r^2 - (3/2r)de^{\frac{1}{2}(\nu-\lambda)}/dr^*, \qquad (15)$$

with $l^{*2} = l(l+1)$, is the effective potential which governs the motion of the gravitational waves of angular momentum number l. This potential is very similar to that governing the motion of electromagnetic waves[1]

$$V_{\text{em}} = e^\nu l^{*2}/r^2$$

and of neutrinos[3]

$$V_\nu = e^\nu l^{*2}/r^2 \mp e^{\nu-\frac{1}{2}\lambda}l^*/r^2 \pm (l^*/r)de^{\frac{1}{2}\nu}/dr^*$$

in a spherically symmetric gravitational field. For *large* l^* these potentials are *identical*.

Trapping of Gravitational Waves

For large l^* Eq. (15), together with the requirement that e^ν be positive throughout, shows that no truly bound states of gravitational waves can exist in any background metric of the type assumed in (2). However, it is possible to construct a metric with an inner region, a barrier region, and an outer region, in such a way that the wave amplitude falls off exponentially in the barrier region. Thus the leakage from the inner region to the outer region is greatly inhibited. Effectively bound states of long life then exist for gravitational waves in the inner region.

Such trapping of gravitational waves is illustrated by the metric of a thin shell spherical (electromagnetic) geon,[1] which may, for the purposes of discussing the response of the waves to the metric, be considered as given and fixed:

$$e^\nu = e^{-\lambda} = 1 - (2GM/c^2r) \quad \text{for} \quad r \geq (9/4)(GM/c^2) = a,$$
$$e^\nu = 1/9 \quad e^{-\lambda} = 1 \quad \text{for} \quad r \leq (9/4)(GM/c^2) = a.$$

The corresponding potential and its properties are illustrated in Fig. 2.

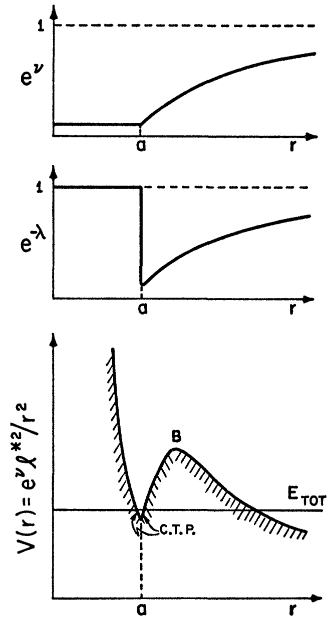

FIG. 2. The upper part of the figure shows the geon metric which results from the calculation of this paper. The gravitational waves are small in amplitude and nonzero only in a thin shell centered at a. The lower part of the figure shows the resulting potential (for large l^*) in which these gravitational waves move. The total energy of the waves, E_{tot}, must be positive, so that it is always energetically possible for the waves to escape to infinity. However, this leakage of radiation is inhibited by the high barrier B. Therefore the radiation can be trapped for a long time in the "active region" bounded by the classical turning points (C.T.P.) for the motion at energy E_{tot}.

IV. GENERAL FEATURES OF A THIN-SHELL SPHERICAL GRAVITATIONAL GEON

Having seen that gravitational radiation can be trapped, we now wish to pose the *geon* problem: Can a trapping metric be produced by the gravitational waves themselves? We do not want to investigate here all possible kinds of spherical gravitational geons, but only analyze a typical, simple representative. Therefore we shall make the same simplifying assumptions as were made for the electromagnetic geon,[1] that the "active region" (the region where $h_{\alpha\beta}$ differ from zero) is a spherical shell, the thickness of which is very small compared to the dimensions of the geon. The metric outside the active region is then already determined up to two constants.

Metric Outside the Active Region

Let the mean radius of the active region be a. For $r > a$ (outside the active region) Eq. (6b) becomes the free-space Einstein equation. We impose the boundary condition that space shall be Euclidean at $r \to \infty$. Then Eq. (6b) has a unique solution for the ν and λ of Eq. (2), the familiar Schwarzschild solution

$$e^\nu = e^{-\lambda} = 1 - (2M/r) \quad \text{for} \quad r > a. \qquad (16)$$

Here M (in cm) $= (G/c^2)\, M$ (in g) is the mass of the geon.

For $r<a$ (outside the active region) the only solution of Eq. (5b) of type (2) which is regular at the origin is flat space-time, $e^\lambda=1$, $e^\nu=\text{const}$. The constant can be determined in terms of the constants M and a characterizing the geon, because e^ν is continuous across an active region of vanishing thickness. This follows from the fact that surfaces S at a constant proper time from a given space-like surface must also be space-like. A discontinuity in e^ν would imply a jump between the part of S at $r<a$ and the part at $r>a$, which could not be bridged by a *space-like* surface *within* the active region. (See Appendix for a more detailed argument.) Comparison with Eq. (16) shows

$$e^\lambda=1\,,\quad e^\nu\equiv b=1-(2M/a)\quad \text{for}\quad r<a. \tag{17}$$

Thus the two constants a, M describe the metric everywhere except in the active region. The constant a can also be determined, so that the final large-scale structure of the geon is described by a single parameter, M. To determine a we must consider the interaction of waves and background *within* the active region ("analysis of stress balance").

V. INTERACTION OF WAVES AND BACKGROUND TREATED VIA VARIATIONAL PRINCIPLE

The equations (5) which determine the interaction of background and waves can be derived from a principle of least action. To find this principle, consider an expansion of the Einstein equations in powers of $h_{\alpha\beta}$. Separately equate to zero the sum of the even terms, and the sum of the odd terms. The equations thus obtained follow from the variational principle

$$\delta\int[L(\gamma_{\alpha\beta}+h_{\alpha\beta})-L(\gamma_{\alpha\beta}-h_{\alpha\beta})]d^4x=0 \tag{18}$$

by independent variation of $\gamma_{\alpha\beta}$ and $h_{\alpha\beta}$. Here $L(m_{\alpha\beta})=R(m_{\alpha\beta})(\det m_{\alpha\beta})^{1/2}$ is the Lagrangian for general relativity.

The equations thus obtained also are identical to second order to the equations (5) without the time average. The latter therefore follow from the terms up to second order of the variational principle (18),

$$\delta\int[L(\gamma_{\alpha\beta})+\Delta_2L(\gamma_{\alpha\beta},h_{\alpha\beta})]d^4x=0. \tag{19}$$

Equations (5b) impose no restriction on the angle or time-dependent parts of $h_{\mu\nu}$ because of the averaging they contain. When the explicit form of the $h_{\mu\nu}$ [Eq. (8)] is substituted into Eq. (19) the action is given by an integral over radial distance only,

$$\delta\int_0^\infty dr[L(\gamma_{\alpha\beta})+\langle\Delta_2L(\gamma_{\alpha\beta},h_{\alpha\beta})\rangle]=0. \tag{20}$$

When varied with respect to $h_{01}(r)$ and $h_{03}(r)$ this action principle yields the radial wave equations corresponding to $\Delta_1R_{01}=0$ and $\Delta_1R_{03}=0$, and variation with respect to $\nu(r)$ and $\lambda(r)$ gives the two independent field equations of Eq. (5b).

Analysis of the Active Region

The integrand of Eq. (20) contains two terms. The first is the well-known Lagrangian density of the gravitational (background) field, and the second describes the interaction of the waves and the background. It is not difficult to find the explicit form of this second term, since it is quadratic in $h_{\alpha\beta}$, and the radial wave equation must follow from it. We could therefore go through an analysis similar to what was done for the electromagnetic geon[1]: (a) approximate the radial wave equations (10) in the limit of a thin active region (large l^*); (b) from these find the interaction term of the action principle in this limit (notation same as in Wheeler[1])

$$\langle\Delta_2L\rangle\sim i[kf_1(df_0/d\rho)-f_0d(kf_1)/d\rho]+jk^2f_1^2+f_0^2; \tag{21}$$

(c) use this Lagrangian to evaluate the terms appearing on the right side of the Einstein equations for the background metric ("effective stress-energy of gravitational waves")

$$G_{\mu\nu}=\frac{\delta L(\gamma)}{\delta\gamma^{\mu\nu}}=-\frac{\delta\langle\Delta_2L\rangle}{\delta\gamma^{\mu\nu}}=-T_{\mu\nu}{}^{\text{eff}} \tag{22}$$

and thus obtain two independent equations for the response of the background to the waves, e.g.,

$$G_{00}=-e^{\nu-\lambda}(\lambda_r/r-1/r^2)-e^\nu/r^2\sim j(kf_1)^2; \tag{23}$$

(d) obtain a simultaneous solution of these equations and the wave equation (14). Since the "effective energy density" T_{00} of the waves is positive definite [right side of Eq. (23)], the M occurring in the metric (16), (17) outside the active region is positive. Therefore the metric will be able to trap the waves, as discussed in Fig. 2 and the Appendix, and a solution for a persistent geon can be obtained.

The details of the active region have only one effect on the large-scale structure of the geon: They determine the value of the constant b in the metric (17). However, it is not *necessary* to analyze the active region in great detail to find the value of this constant.[2] We need only one consequence of the complete set of equations, (14), (23), which we shall now derive using the equivalent variational principle (20).

In the variational principle which yields the radial wave equation

$$\delta_h\int\Delta_2L(\gamma_{\alpha\beta},h_{\alpha\beta})d^4x=0 \tag{24}$$

consider the particular variation $h_{\alpha\beta}\to(1+\epsilon)h_{\alpha\beta}$, with $\epsilon=\text{const}$. If $h_{\alpha\beta}$ is a solution of the wave equation, (24) must hold. Also, since δ_2L is quadratic in the h's, the variation can be written as

$$\epsilon\frac{\partial}{\partial\epsilon}\int(1+\epsilon)^2\Delta_2L(\gamma_{\alpha\beta},h_{\alpha\beta})d^4x=0$$

giving

$$\int \Delta_2 L d^4x = \int_{\substack{\text{active}\\ \text{region}}} \langle \Delta_2 L \rangle dr = 0.$$

Compute

$$\Delta_2 L = \Delta_2[g^{\alpha\beta}R_{\alpha\beta}(-g)^{1/2}] = \gamma^{\alpha\beta}\Delta_2 R_{\alpha\beta}(-\gamma)^{1/2} + \Delta_1[\gamma^{\alpha\beta}(-g)^{1/2}]\Delta_1 R_{\alpha\beta} + R_{\alpha\beta}\Delta_2[\gamma^{\alpha\beta}(-g)^{1/2}],$$

and note that by Eqs. (6) the last two terms vanish or are of fourth order in h. Therefore we have, using Eq. (6b) again,

$$\langle \Delta_2 L \rangle = \gamma^{\alpha\beta}\langle \Delta_2 R \rangle (-\gamma)^{1/2} = \gamma^{\alpha\beta}R_{\alpha\beta}(-\gamma)^{1/2} = R(-\gamma)^{1/2},$$

and we find

$$\int_{\substack{\text{active}\\ \text{region}}} R(\gamma_{\alpha\beta})(-\gamma)^{1/2} dr = 0. \tag{25}$$

To find the constant b we substitute the explicit expression[11] for $R(\gamma_{\alpha\beta})$ in terms of $\nu(r)$ and $\lambda(r)$ and carry out the integration over the active region. We assume the width of the active region small enough that we may set $r=a$ everywhere. The limits of the active region are denoted by $-$ and $+$:

$$\begin{aligned} O &= \int_-^+ R(\gamma_{\alpha\beta})(-\gamma)^{1/2} dr = \int_-^+ [d(\nu_r e^{\frac{1}{2}(\nu-\lambda)})/dr \\ &\quad + 4d(e^{\frac{1}{2}(\nu-\lambda)})/rdr + 2e^{\frac{1}{2}(\nu-\lambda)}/r^2 - 2e^{\frac{1}{2}(\nu+\lambda)}/r^2]r^2 dr \\ &= a^2[\nu_r e^{\frac{1}{2}(\nu-\lambda)} + 4e^{\frac{1}{2}(\nu-\lambda)}/a]_-^+ \\ &\quad + 2\int_-^+ (e^{\frac{1}{2}(\nu-\lambda)} - e^{\frac{1}{2}(\nu+\lambda)}) dr. \end{aligned} \tag{26}$$

In the limit of a thin active region the last integral contributes nothing. From Eqs. (16), (17) we have

$$\begin{gathered} \nu_r(-)=0, \quad e^{\lambda(-)}=1, \quad e^{-\lambda(+)}=e^{\nu(+)}=e^{\nu(-)}=b, \\ \nu_r(+)=(1-b)/ab. \end{gathered} \tag{27}$$

Insert in Eq. (26) and solve the resulting quadratic for b. One solution, $b=1$, corresponds to the trivial case of flat space. The other solution is the one appropriate for the geon:

$$b = 1/9, \tag{28}$$

so that from (17)

$$a = (9/4)M. \tag{29}$$

The large scale features of the geon metric are now completely determined by the one number M. In turn, this can be expressed through the parameters l^*, ω, which specify the gravitational waves, in the following way. If the active region is thin, the inner classical turning point in the potential (15) is a measure of the position of the active region:

$$\omega^2 - bl^{*2}/a^2 = 0.$$

[11] See, for example, Ref. 7, p. 242.

Thus we find

$$\begin{aligned} M &= 4l^*/27\omega, \\ a &= l^*/3\omega. \end{aligned} \tag{30}$$

The procedure leading to Eqs. (28) and (29) has the simple physical significance of a balance of stresses: In an equilibrium situation, the radial gravitational pull must be balanced by the transverse pressure of the radiation. The situation is analogous for all zero-rest-mass equation fields. In fact, *the procedure of Eqs. (26) and (27) and its results (28) and (29) apply to thin-shell spherical geons constructed from any zero-rest-mass field obeying a linear field equation.* For zero-rest-mass fields other than gravity, this follows from the result of Buchdahl[12] that the trace of the stress energy tensor vanishes,[13] which implies (25).

VI. CONCLUSION

It is not surprising that the large-scale features of electromagnetic, gravitational, and other zero-rest-mass field geons should agree very closely. For large azimuthal index (l^*) the trapped radiation can be treated in the "classical limit," as rays travelling along null geodesics. For nearly circular null geodesics in the metric (2) the *geodesic law* shows that the behavior of the rays is governed by the quantity e^ν/r^2. For a metric of the form (16), (17) this quantity has the shape of a potential well near $r=a$ (see, for example, Fig. 2). In the classical limit the intensity distribution over the active region will therefore be alike for the two kinds of geon, provided that the background metrics are the same. Conversely, the same intensity distribution would be expected to lead to the same background metrics. The detailed procedure used to investigate the gravitational geon are in many ways analogous to those used for its electromagnetic counterpart as is displayed in Table I.

Two important features of gravitational waves are brought to light by our analysis of the gravitational geon: (a) The propagation of gravitational waves even in strongly curved spaces need not always be treated through the full nonlinear Einstein equations. A linear approximation can be justified even when the waves carry a substantial *energy density* provided only that their *amplitude* is small. (This method of analysis has also been applied to the dynamics of closed model universes filled with gravitational radiation.[14]) (b) Gupta,[15] Feynman,[16] and Thirring[17] have focused attention on the regenerative feature of gravitation, that

[12] H. Buchdahl, J. Australian Math. Soc. **1**, 99 (1959).

[13] The zero-rest-mass scalar field is an exception to Buchdahl's result. However, by explicit calculation it is readily seen that the integrated form, Eq. (25), holds also for this case, as a consequence of the field equations.

[14] D. R. Brill, Nuovo Cimento Suppl. (to be published).

[15] S. N. Gupta, Rev. Mod. Phys. **29**, 334 (1957).

[16] R. Feynman, Lectures at the California Institute of Technology 1962–63 (unpublished).

[17] W. Thirring, Fortschritte Phys. **7**, 79 (1959); Ann. Phys. (N. Y.) **16**, 96 (1961).

TABLE I. Spherical electromagnetic and gravitational geons.

	Electromagnetic	Gravitational
Source of energy producing curvature of space time	Electromagnetic waves $F_{\mu\nu}=A_{\mu,\nu}-A_{\nu,\mu}$	Gravitational waves $h_{\mu\nu}$
Linear wave equation which governs propagation of waves in background metric	Maxwell's Equations $g^{\alpha\beta}A_{\mu;\alpha\beta}=0$	$\Delta_1 R_{\mu\nu}=0$
Infinitesimal gauge transformation which leaves the physics unchanged	$A_\mu'=A_\mu+\Lambda_{,\mu}$	$h_{\mu\nu}'=h_{\mu\nu}+\xi_{\mu;\nu}+\xi_{\nu;\mu}$
Stress energy of field	$T_{\mu\nu}{}^{\mathrm{em}}=F_{\mu\sigma}F^{\sigma}{}_{\nu}-\frac{1}{4}g_{\mu\nu}F_{\alpha\beta}F^{\alpha\beta}$	$-\Delta_2 R_{\mu\nu}$
Gravitational field equations	$R_{\mu\nu}(\gamma_{\alpha\beta})=\langle T_{\mu\nu}{}^{\mathrm{em}}\rangle$	$R_{\mu\nu}(\gamma_{\alpha\beta})=-\langle\Delta_2 R_{\mu\nu}(\gamma_{\alpha\beta},h_{\alpha\beta})\rangle$
Lagrangian for field equations and wave equation	$L(\gamma_{\alpha\beta})+\langle(-\gamma)^{1/2}F_{\alpha\beta}F^{\alpha\beta}\rangle$ vary $\gamma_{\alpha\beta}$ and $F_{\mu\nu}$	$L(\gamma_{\alpha\beta})+\langle\Delta_2 L(\gamma_{\alpha\beta},h_{\alpha\beta})\rangle$ vary $\gamma_{\alpha\beta}$ and $h_{\alpha\beta}$
Radial wave equation in the large angular momentum approximation	$(d^2R/dr^{*2})+[k^2-e^\nu l^{*2}/r^2]R=0$ $R=$radial part of A_φ	$d^2Q/dr^2+[k^2-(e^\nu l^{*2}/r^2)]Q=0$ $Q=$radial part of h_{03}
Equation from which large scale metric can be derived	$R(g_{\alpha\beta})=0$ $\int_{\substack{\text{active}\\\text{region}}} R(g_{\alpha\beta})(-g)^{1/2}d^4x=0$	$\int_{\substack{\text{active}\\\text{region}}} R(\gamma_{\alpha\beta})(-\gamma)^{1/2}d^4x=0$
Large scale metric which results	$e^{-\lambda}=e^\nu=1-2M/r \quad r>a$; $e^{-\lambda}=1,\quad e^\nu=1/9 \quad r<a$; $a=1/3(l^*/k) \quad M=4/27(l^*/k)$	

the fields are their own source, as distinct in principle from the electromagnetic and other fields. In the present work, however, the focus of attention is the reality of the effective stress energy of gravitational radiation as shown especially by the fact that it and it alone is responsible for the entire mass of the type of geon discussed here.

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APPENDIX

The continuity of e^ν across the active region follows from the radial wave equation (14) when the thickness of the geon's active region is small compared to its dimensions. Let us determine the criteria for this result.

For the electromagnetic geon, Wheeler[1] has estimated the width δa and the radius of the active region from a radial wave equation which is identical in form (for large l^*) with that of the gravitational case (14). His analysis can, therefore, be taken over directly:

$$\delta a/a\sim l^{*-2/3},\quad \omega a\sim l^*,\quad \omega M\sim l^*. \tag{A1}$$

Therefore the width of the active region is small compared to the geon's radius when l^* is large compared to unity.

From (A1) Wheeler derives the following expansion of r about the active region:

$$\omega r=a+l^{*1/3}q+\cdots. \tag{A2}$$

Now ask for the behavior of λ_r and ν_r as the width of the active region decreases (l^* increasing) while holding the mass and radius fixed by requiring $\omega\sim l^*$. We already know from Eqs. (16), (17) that e^λ is discontinuous so that λ_r must become infinite as $\delta a/a\to 0$, and indeed since $e^\lambda\sim(l^*)^0$ we find

$$\lambda_r\sim l^{*2/3}. \tag{A3}$$

To deduce the behavior of ν_r we note that the radial wave equation (14) implies

$$(\omega^2-e^\nu l^{*2}/r^2)\sim l^{*4/3}. \tag{A4}$$

Thus,

$$e^\nu=a+l^{*-2/3}A(q)+\cdots \tag{A5}$$

and

$$\nu_r\sim(l^*)^0.$$

Since ν_r remains finite as $\delta a/a\to 0$, e^ν must be continuous across an active region of vanishing thickness.